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On extended Taub–NUT metrics

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Much attention has been paid to the (Euclidean) Taub–NUT metric because the geodesic on this space describes approximately the motion of two well-separated interacting monopoles. It is also well known that the Taub–NUT metric admits a Kepler-type symmetry. In this paper, the Taub–NUT metric is extended so that it still admits a Kepler-type symmetry. The geodesics of this metric will be investigated. In particular, regularization of singular geodesics is studied by use of a method from dynamical systems. Further, some geometrical properties of the extended Taub–NUT metric are cleared up. In order that the extended Taub–NUT metric either has a self-dual Riemann curvature tensor or is an Einstein metric, it is necessary and sufficient that it coincides with the original Taub–NUT metric up to a constant factor. Furthermore, a class of extended Taub–NUT metrics which have a self-dual Weyl curvature tensor is found. This class of metrics, of course, includes the Taub–NUT metric.

Keywords: *extended Taub–NUT metric, Kepler-type symmetry, self-dual and conformally self-dual curvature tensors*
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1. Introduction

Much attention has been paid to the Euclidean Taub–NUT metric, since in the long-distance limit the relative motion of two monopoles is described approximately by its geodesics [1,2]. From the symmetry viewpoint, of particular interest is the fact that the geodesic motion admits a Kepler-type symmetry, if a cyclic variable is gotten rid of [3,4]. L.Gy. Fehér and P.A. Horváthy investigated the same symmetry [5], and soon extended the symmetry to the o(4, 2) dynamical symmetry well known for the Kepler problem, in a paper co-authored with B. Cordani [6]. Those authors presented a general framework for understanding Kepler-type symmetries [7]. See also ref. [8] for a review of the dynamical symmetry of monopole scattering.

A generalization of the Euclidean Taub–NUT metric is to be discussed in the form

$$\mathrm{d}s_{\mathrm{G}}^2 = f(r)(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)) + g(r)(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2, \qquad (1.1)$$

where $\sqrt{r}$, $r > 0$, is the radial coordinate of $\mathbb{R}^4 - \{0\}$ and the angle variables (θ, ϕ, ψ) $(0 \leq \theta \leq \pi,\ 0 \leq \phi < 2\pi,\ 0 \leq \psi < 4\pi)$ parametrize the unit three-sphere S^3, and $f(r)$ and $g(r)$ are arbitrary functions in r. If $f(r)$ and $g(r)$ are taken to be

$$f(r) = 1 + \frac{4m}{r}, \qquad g(r) = \frac{(4m)^2}{1 + 4m/r}, \qquad (1.2)$$

respectively, the metric $\mathrm{d}s_{\mathrm{G}}^2$ becomes the Euclidean Taub–NUT metric with $m = -1/2$. For $m > 0$, the metric (1.1) with (1.2) is just the space part of the metric of the celebrated Kaluza–Klein monopole of Gross and Perry and of Sorkin [9]. In this article, by the Taub–NUT metric simply the metric (1.1) with (1.2) is meant. Interest will center on those metrics $\mathrm{d}s_{\mathrm{G}}^2$ that are generalized so that they may admit the same Kepler-type symmetry as the Taub–NUT metric does when the cyclic variable ψ is gotten rid of. In this paper, those metrics will be referred to as the extended Taub–NUT metrics and denoted by $\mathrm{d}s_{\mathrm{K}}^2$ because of the Kepler-type symmetry. It will be shown that for $\mathrm{d}s_{\mathrm{K}}^2$ the functions $f(r)$ and $g(r)$ take, respectively, the form

$$f(r) = \frac{a}{r} + b, \qquad g(r) = \frac{ar + br^2}{1 + cr + dr^2}, \qquad (1.3)$$

where a, b, c, d are constants. If the constants are subject to the constraints

$$c = 2b/a, \qquad d = (b/a)^2, \qquad (1.4)$$

the extended metric coincides, up to a constant factor, with the original Taub–NUT metric on setting $4m = a/b$.

This paper has two aims. One is to derive the extended Taub–NUT metric $\mathrm{d}s_{\mathrm{K}}^2$ and to study its geodesics from the viewpoint of dynamical systems. The other aim is to investigate geometrical properties of the extended Taub–NUT metric $\mathrm{d}s_{\mathrm{K}}^2$, say, to ask when it is Einstein, when it has a self-dual Riemann curvature form, and when it has a self-dual Weyl curvature tensor. These questions are raised in order to observe to what extent the metric $\mathrm{d}s_{\mathrm{K}}^2$ is actually extended. In fact, the Taub–NUT metric has received interest as a self-dual Einstein metric [10].

The organization of this article is as follows: Section 2 is concerned with setting up the generalized Taub–NUT metric $\mathrm{d}s_{\mathrm{G}}^2$ and contains a few observations on the metric. In fact, the components of the curvature forms are written out by use of an orthonormal basis together with the functions $f(r)$ and $g(r)$ included in the metric. The components of the Ricci tensor are written out as well in terms of $f(r)$ and $g(r)$. By using the expressions obtained, some observations are made on $\mathrm{d}s_{\mathrm{G}}^2$; the functions $f(r)$ and $g(r)$ are determined so that the metric $\mathrm{d}s_{\mathrm{G}}^2$ may be flat, conformally flat, or Einstein in addition to being conformally flat.

Section 3 contains the derivation of extended Taub–NUT metrics from the viewpoint of dynamical systems. The generalized Taub–NUT metric ds_G^2 determines a geodesic flow system on the cotangent bundle $T^*(\mathbb{R}^4 - \{0\})$, which system can be reduced to a system on $T^*(\mathbb{R}^3 - \{0\})$ by getting rid of a cyclic variable. The reduced system admits manifest rotational invariance, and hence has a conserved angular momentum. With the assumption of the existence of an additional conserved vector, a Rünge–Lenz-type vector, for the reduced system, the functions $f(r)$ and $g(r)$ are to be determined to bring about the extended Taub–NUT metric ds_K^2. Trajectories of the reduced system from ds_K^2 are then discussed by use of the conservation of the angular momentum and the Runge–Lenz vector. It turns out that generic trajectories are conic sections. However, since the metric could have a singularity, singular trajectories could possibly occur. By using the conservation law, regularization of singular trajectories is performed, which, in turn, becomes regularization of singular geodesics for ds_K^2 as it is. On the basis of trajectories analyzed in the reduced system, the geodesic equation for ds_K^2 is dealt with to look for closed geodesics.

Section 4 is concerned with the Riemann curvature tensor for ds_K^2. It will be shown that the constraints (1.4) are necessary and sufficient conditions for the extended Taub–NUT metric ds_K^2 either to be an Einstein metric or to have a self-dual Riemann curvature tensor. As was anticipated in the sentence about (1.4), the metric ds_K^2 in this case coincides with the Taub–NUT metric up to a constant factor. Hence the self-duality of the Taub–NUT metric [10] is verified as a byproduct.

Section 5 contains the investigation of a necessary and sufficient condition for the metric ds_K^2 to have a self-dual Weyl tensor, or to be conformally self-dual. It will turn out that either the metric ds_K^2 with $2 + cr > 0$ is conformally self-dual or the metric ds_K^2 with $2 + cr < 0$ is conformally anti-self-dual, if and only if $d = c^2/4$. This class of metrics contains, of course, the class of those studied in section 4.

In the present paper, the Kepler-type symmetry is a guiding principle of the generalization of the Taub–NUT metric. Because of the conserved angular momentum and Runge–Lenz vectors, all the bounded trajectories for the reduced system are closed. One can then take another approach to the generalization of the Taub–NUT metric from the viewpoint of closed trajectories. In fact, one may ask when all the bounded trajectories for the reduced system are closed. It will be shown in another paper that there are two types of functions f and g and therefore two types of such metrics. One is the extended Taub–NUT metric ds_K^2 mentioned above, which has a resemblance to the Kepler problem in a way. The other is determined by the functions

$$f(r) = ar^2 + b, \qquad g(r) = \frac{ar^4 + br^2}{1 + cr^2 + dr^4}, \tag{1.5}$$

where a, b, c, d are constants. In contrast with the metric ds_K^2, these metrics will be called harmonic oscillator type metrics for the reason that the reduced system has a marked resemblance to the harmonic oscillator. Details will appear elsewhere [11].

2. Setting up the structure equations

In this section, we are to treat generalized Taub–NUT metrics and to write out their Riemann curvature tensor. To start with, we note that the curvilinear coordinates (r, θ, ϕ, ψ) are related to the Cartesian coordinates (y^1, y^2, y^3, y^4) in $\mathbb{R}^4$ by

$$\begin{aligned} y^1 &= \sqrt{r}\cos\tfrac{1}{2}(\psi+\phi)\cos\tfrac{1}{2}\theta, & y^2 &= \sqrt{r}\sin\tfrac{1}{2}(\psi+\phi)\cos\tfrac{1}{2}\theta, \\ y^3 &= \sqrt{r}\cos\tfrac{1}{2}(\psi-\phi)\sin\tfrac{1}{2}\theta, & y^4 &= \sqrt{r}\sin\tfrac{1}{2}(\psi-\phi)\sin\tfrac{1}{2}\theta. \end{aligned} \tag{2.1}$$

The ranges of variables are the same as stated in the introduction. The generalized Taub–NUT metric we are to treat, ds_G^2, is expressed as in (1.1). It is convenient for us to introduce orthonormal one-forms ω^i defined by

$$\begin{aligned} \omega^1 &= f(r)^{1/2}\,dr, \\ \omega^2 &= r f(r)^{1/2}(-\sin\psi\, d\theta + \cos\psi\sin\theta\, d\phi), \\ \omega^3 &= r f(r)^{1/2}(\cos\psi\, d\theta + \sin\psi\sin\theta\, d\phi), \\ \omega^4 &= g(r)^{1/2}(d\psi + \cos\theta\, d\phi). \end{aligned} \tag{2.2}$$

Clearly, one has $ds_G^2 = \sum_{i=1}^4 (\omega^i)^2$.

The setting of our generalized Taub–NUT metric is in keeping with the bundle structure of the space $\mathbb{R}^4 - \{0\}$: $U(1) \to \mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$. In fact, the action of the structure group $U(1)$ is expressed as $\psi \to \psi + t$ with the other variables fixed, and has the infinitesimal generator $\partial/\partial\psi$. A dual form $d\psi + \cos\theta\, d\phi$ to $\partial/\partial\psi$ is the connection form for the natural connection defined on that bundle. ω^4 is a multiple of the connection form, a vertical form, and the forms ω^1, ω^2, and ω^3 are horizontal to the $U(1)$ action. In a dual manner, one can introduce orthonormal vector fields e_i dual to ω^i,

$$\begin{aligned} e_1 &= \frac{1}{f(r)^{1/2}}\frac{\partial}{\partial r}, \\ e_2 &= \frac{1}{r f(r)^{1/2}}\left(-\sin\psi\frac{\partial}{\partial\theta} + \cos\psi\csc\theta\frac{\partial}{\partial\phi} - \cos\psi\cot\theta\frac{\partial}{\partial\psi}\right), \\ e_3 &= \frac{1}{r f(r)^{1/2}}\left(\cos\psi\frac{\partial}{\partial\theta} + \sin\psi\csc\theta\frac{\partial}{\partial\phi} - \sin\psi\cot\theta\frac{\partial}{\partial\psi}\right), \\ e_4 &= \frac{1}{g(r)^{1/2}}\frac{\partial}{\partial\psi}. \end{aligned} \tag{2.3}$$

The vector fields e_1, e_2, e_3 and the e_4 are horizontal and vertical, respectively, with respect to the natural connection stated above. Further, we notice that the coordinates (r, θ, ϕ), when projected on the base space $\mathbb{R}^3 - \{0\}$, can be looked upon as spherical coordinates.

2.1. THE CONNECTION AND CURVATURE FORMS

We are now ready to consider the Levi-Civita connection and its curvature forms. The covariant derivative of the frame e_i is put in the form

$$\mathrm{d}^\nabla e_i = \sum_{j=1}^{4} \omega_i^j e_j, \tag{2.4}$$

where ω_i^j are one-forms called connection forms with respect to the Levi-Civita connection. One of Cartan's structure equations is then expressed as

$$\mathrm{d}\omega^i + \sum_j \omega_j^i \wedge \omega^j = 0. \tag{2.5}$$

A lengthy calculation provides the connection forms as follows:

$$(\omega_i^j) = \begin{pmatrix} 0 & -A(r)\omega^2 & -A(r)\omega^3 & -B(r)\omega^4 \\ A(r)\omega^2 & 0 & -C(r)\omega^4 & -D(r)\omega^3 \\ A(r)\omega^3 & C(r)\omega^4 & 0 & D(r)\omega^2 \\ B(r)\omega^4 & D(r)\omega^3 & -D(r)\omega^2 & 0 \end{pmatrix}, \tag{2.6}$$

where the functions $A(r)$ to $D(r)$ are given by

$$A(r) = \frac{\mathrm{d}(r^2 f(r))/\mathrm{d}r}{2\cancel{r^2 f(r) f(r)^{1/}}2},$$

$$B(r) = \frac{\mathrm{d}g(r)/\mathrm{d}r}{2\cancel{g(r)f(r)}^{1/2}},$$

$$C(r) = \frac{g(r) - 2r^2 f(r)}{2r^2 f(r) g(r)^{1/2}},$$

$$D(r) = \frac{g(r)^{1/2}}{2r^2 f(r)}. \tag{2.7}$$

The other structure equation of Cartan gives the curvature forms Ω_i^j,

$$\Omega_i^j = \mathrm{d}\omega_i^j + \sum_k \omega_k^j \wedge \omega_i^k. \tag{2.8}$$

A straightforward calculation results in

$$\begin{aligned} \Omega_1^2 = {} & [A(r)^2 + A'(r) f(r)^{-1/2}]\omega^1 \wedge \omega^2 \\ & + D(r)[A(r) - B(r)]\omega^3 \wedge \omega^4, \end{aligned}$$

$$\begin{aligned}
\Omega_1^3 &= [A(r)^2 + A'(r)f(r)^{-1/2}]\omega^1 \wedge \omega^3 \\
&\quad -D(r)[A(r) - B(r)]\omega^2 \wedge \omega^4, \\
\Omega_1^4 &= [B(r)^2 + B'(r)f(r)^{-1/2}]\omega^1 \wedge \omega^4 \\
&\quad -2D(r)[A(r) - B(r)]\omega^2 \wedge \omega^3, \\
\Omega_2^3 &= [B(r)C(r) + C'(r)f(r)^{-1/2}]\omega^1 \wedge \omega^4 \\
&\quad +[2C(r)D(r) + A(r)^2 + D(r)^2]\omega^2 \wedge \omega^3, \\
\Omega_2^4 &= [A(r)D(r) + D'(r)f(r)^{-1/2}]\omega^1 \wedge \omega^3 \\
&\quad +[A(r)B(r) - D(r)^2]\omega^2 \wedge \omega^4, \\
\Omega_3^4 &= -[A(r)D(r) + D'(r)f(r)^{-1/2}]\omega^1 \wedge \omega^2 \\
&\quad +[A(r)B(r) - D(r)^2]\omega^3 \wedge \omega^4,
\end{aligned} \tag{2.9}$$

where the prime denotes the derivative with respect to r, and the other components are known from the anti-symmetry $\Omega_i^j + \Omega_j^i = 0$. The Riemann curvature tensor has the components $R^j_{ik\ell} := \Omega_i^j(e_k, e_\ell)$. As it would take a page to list the explicit form of $R^j_{ik\ell}$, we proceed, instead, to calculate the Ricci tensor with components R_{ij} and the scalar curvature R. They are given as follows:

$$\begin{aligned}
R_{11} &= -2[A(r)^2 + A'(r)f(r)^{-1/2}] - [B(r)^2 + B'(r)f(r)^{-1/2}], \\
R_{22} &= -2A(r)^2 - A'(r)f(r)^{-1/2} - 2C(r)D(r) - A(r)B(r), \\
R_{33} &= -2A(r)^2 - A'(r)f(r)^{-1/2} - 2C(r)D(r) - A(r)B(r), \\
R_{44} &= -B(r)^2 - B'(r)f(r)^{-1/2} + 2D(r)^2 - 2A(r)B(r), \\
R_{ij} &= 0 \quad \text{for } i \neq j,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
R &= -6A(r)^2 - 4A'(r)f(r)^{-1/2} - 2B(r)^2 - 2B'(r)f(r)^{-1/2} \\
&\quad -4C(r)D(r) - 4A(r)B(r) + 2D(r)^2.
\end{aligned} \tag{2.11}$$

2.2. EASY OBSERVATIONS ON THE GENERALIZED TAUB–NUT METRIC

We here make a few observations on the generalized Taub–NUT metric, using the expression obtained in the last subsection. First we give the expression of the standard flat metric ds_0^2,

$$\begin{aligned}
ds_0^2 &= \textstyle\sum_{i=1}^4 (dy^i)^2 \\
&= \frac{1}{4r}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2 + (d\psi + \cos\theta\, d\phi)^2)].
\end{aligned} \tag{2.12}$$

We start with the following proposition.

Proposition 2.1. *In order that the generalized Taub–NUT metric* ds_G^2 *be flat, it is* *necessary and sufficient that*

$$f(r) = \frac{\alpha}{r}, \qquad g(r) = r^2 f(r) = \alpha r, \quad \alpha > 0\text{: const.,} \tag{2.13a}$$

or

$$f(r) = \frac{\alpha}{r^3}, \qquad g(r) = r^2 f(r) = \frac{\alpha}{r}, \quad \alpha > 0\text{: const.} \tag{2.13b}$$

Proof. The condition $R^i_{jk\ell} = 0$ yields the following simultaneous differential and functional equations for $f(r)$ and $g(r)$, after a long calculation along with (2.9):

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^2 f(r)) = \pm r f(r), \qquad g(r) = r^2 f(r). \tag{2.14}$$

These are easily solved to give (2.13a) and (2.13b), as desired. □

It is to be noted that the metric corresponding to (2.13a) is a constant multiple of the standard flat metric (2.12). On the other hand, the metric corresponding to (2.13b) can be transformed into the standard flat metric through the coordinate transformation $R_0 = 2(\alpha/r)^{1/2}$, where R_0 is the usual radial variable in $\mathbb{R}^4$.

Further, comparison of the generalized Taub–NUT metric (1.1) with the standard flat metric (2.12) provides the following proposition.

Proposition 2.2. *The generalized Taub–NUT metric* $\mathrm{d}s^2_{\mathrm{G}}$ *is conformally flat, if and only if*

$$g(r) = r^2 f(r). \tag{2.15}$$

This proposition can also be proved by showing that the conformal curvature tensor with components $C^i_{jk\ell}$ defined by

$$C^i_{jk\ell} = R^i_{jk\ell} - \tfrac{1}{2}(\delta^i_k R_{j\ell} - \delta^i_\ell R_{jk} + \delta^j_\ell R_{ik} - \delta^j_k R_{i\ell}) + \tfrac{1}{6} R(\delta^i_k \delta_{j\ell} - \delta^i_\ell \delta_{jk}) \tag{2.16}$$

vanishes if and only if $g(r) = r^2 f(r)$. We will come back to this question in section 5.

In addition, we can prove the following theorem under the condition that $\mathrm{d}s^2_{\mathrm{G}}$ is conformally flat.

Theorem 2.3. *In order that the conformally flat generalized Taub–NUT metric be an Einstein metric, it is necessary and sufficient that*

$$f(r) = \frac{\alpha}{r(\beta + \gamma r)^2}, \qquad g(r) = r^2 f(r), \quad \alpha > 0,\ \beta,\ \gamma\text{: consts.} \tag{2.17}$$

Proof. Before proving this, we have to remark that (2.17) reduces to (2.13a) or (2.13b), according to whether γ or β is taken to be zero. With the assumption

$g(r) = r^2 f(r)$, the functions (2.7) are put in the simpler form

$$A(r) = B(r) = \frac{\mathrm{d}(r^2 f(r))/\mathrm{d}r}{2r^2 f(r) f(r)^{1/2}},$$

$$C(r) = -D(r) = -\frac{1}{2rf(r)^{1/2}}. \tag{2.18}$$

Then the Ricci tensor given in (2.10) turns into

$$R_{11} = -3(A(r)^2 + A'(r)f(r)^{-1/2}),$$

$$R_{22} = R_{33} = R_{44} = -3A(r)^2 - A'(r)f(r)^{-1/2} + 2D(r)^2. \tag{2.19}$$

From this it follows that the condition for the metric under consideration to be an Einstein metric, $R_{ij} = \frac{1}{4}R\delta_{ij}$, results in the condition

$$A'(r) = -D(r)^2 f(r)^{1/2}. \tag{2.20}$$

From (2.18) and (2.20), we obtain the differential equation for $f(r)$,

$$3f(r)^2 + 2rf(r)f'(r) + 3r^2(f'(r))^2 - 2r^2 f(r)f''(r) = 0. \tag{2.21}$$

Setting $h(r) := \log f(r)$, one finds that the derivative $h'(r)$ is subject to the Riccati equation

$$2\frac{\mathrm{d}^2 h}{\mathrm{d}r^2} - \left(\frac{\mathrm{d}h}{\mathrm{d}r}\right)^2 - \frac{2}{r}\frac{\mathrm{d}h}{\mathrm{d}r} - \frac{3}{r^2} = 0,$$

which is easily integrated, and eventually one obtains (2.17), a solution to (2.21). Conversely, from (2.17) to (2.19), one has immediately the equation

$$R_{ij} = \frac{3\beta\gamma}{\alpha}\delta_{ij}. \tag{2.22}$$

This completes the proof. □

Before ending this section, we should point out that the Einstein metric derived above is, furthermore, of constant curvature. In fact, as is well known, if a Riemannian space M is conformally flat and Einstein ($\dim M \geq 3$), it is of constant curvature. In our case, the curvature tensor is shown to satisfy the following equation:

$$R^j_{ik\ell} = \frac{\beta\gamma}{\alpha}(\delta^j_k \delta_{i\ell} - \delta^j_\ell \delta_{ik}). \tag{2.23}$$

3. Extended Taub–NUT metrics

In this section, we are to determine the functions $f(r)$ and $g(r)$ contained in $\mathrm{d}s^2_{\mathrm{G}}$ from the viewpoint of dynamical systems. To start with, we consider geodesic flows of the generalized Taub–NUT metric $\mathrm{d}s^2_{\mathrm{G}}$, which has the Lagrangian L on the tangent bundle $T(\mathbb{R}^4 - \{0\})$,

$$L = \tfrac{1}{2}f(r)[\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)] + \tfrac{1}{2}g(r)(\dot{\psi} + \cos\theta\,\dot{\phi})^2, \tag{3.1}$$

where $(\dot{r}, \dot{\theta}, \dot{\phi}, \dot{\psi}, r, \theta, \phi, \psi)$ stand for coordinates in the tangent bundle. The conserved quantity for the cyclic variable ψ is given by

$$\mu = g(r)(\dot{\psi} + \cos\theta\,\dot{\phi}). \tag{3.2}$$

Under this condition, the dynamical system for the geodesic flow on $T(\mathbb{R}^4 - \{0\})$ can be reduced to a system on $T(\mathbb{R}^3 - \{0\})$. This process is in accordance with the bundle structure $\mathbb{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$, since the structure group action is expressed as $\psi \to \psi + t$ with the other variables fixed.

In the Hamiltonian formalism, the reduced Hamiltonian system can be described on the reduced phase space $T^*(\mathbb{R}^3 - \{0\})$ together with the symplectic form ω and the reduced Hamiltonian function, which are given, respectively, by

$$\omega = \sum_{\lambda=1}^{3} \mathrm{d}p_\lambda \wedge \mathrm{d}x^\lambda - \frac{\mu}{2r^3} \sum_{\eta,\lambda,\nu} \varepsilon_{\eta\lambda\nu} x^\eta\, \mathrm{d}x^\lambda \wedge \mathrm{d}x^\nu, \tag{3.3}$$

$$H = \frac{1}{2f(r)} \sum_{\lambda=1}^{3} p_\lambda^2 + \frac{\mu^2}{2g(r)}, \tag{3.4}$$

where $\boldsymbol{x} = (x^\lambda)$ and $\boldsymbol{p} = (p_\lambda)$ are vectors in the factor spaces $\mathbb{R}^3 - \{0\}$ and $\mathbb{R}^3$, respectively, of the reduced phase space $T^*(\mathbb{R}^3 - \{0\}) \cong (\mathbb{R}^3 - \{0\}) \times \mathbb{R}^3$, and $\varepsilon_{\eta\lambda\nu}$ denotes the Levi-Civita symbol with indices ranging over 1,2,3. See, for example, ref. [12] for the reduction. As is seen in (3.3), the reduced symplectic form ω contains the two-form representing a monopole field with strength μ. The equation of motion is determined through the Hamiltonian vector field X_{H} defined by $\iota(X_{\mathrm{H}})\omega = -\mathrm{d}H$, where ι denotes the interior product. After a calculation, we find the Hamiltonian vector field X_{H}, and thereby the equation of motion in the form

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \frac{\boldsymbol{p}}{f(r)}, \\ \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= \frac{f'(r)|\boldsymbol{p}|^2}{2rf(r)^2}\boldsymbol{x} + \mu^2 \frac{g'(r)}{2rg(r)^2}\boldsymbol{x} - \frac{\mu}{r^3 f(r)}\boldsymbol{p} \times \boldsymbol{x}. \end{aligned} \tag{3.5}$$

Owing to the obvious spherical symmetry, we can easily show that the angular momentum vector

$$\boldsymbol{J} = \boldsymbol{x} \times \boldsymbol{p} + (\mu/r)\boldsymbol{x} \tag{3.6}$$

is a conserved vector. In fact, for an arbitrary infinitesimal rotation

$$X = (\boldsymbol{\xi} \times \boldsymbol{x}) \cdot \partial/\partial \boldsymbol{x} + (\boldsymbol{\xi} \times \boldsymbol{p}) \cdot \partial/\partial \boldsymbol{p}, \quad \boldsymbol{\xi} \in \mathbb{R}^3, \tag{3.7}$$

one has

$$\iota(X)\omega = -\boldsymbol{\xi} \cdot \mathrm{d}\boldsymbol{J}. \tag{3.8}$$

3.1. THE DERIVATION OF EXTENDED TAUB–NUT METRICS

We assume here that in addition to the angular momentum vector there exists a conserved vector $\boldsymbol{R}$ of the following form with an unknown constant κ:

$$\boldsymbol{R} = \boldsymbol{p} \times \boldsymbol{J} - (\kappa/r)\boldsymbol{x}. \tag{3.9}$$

This is an analog of the Runge–Lenz vector well known for the Kepler problem. It is also well known that the reduced system from the geodesic flow system for the Taub–NUT metric admits a Kepler-type symmetry [3–6]. We wish to determine the functions $f(r)$ and $g(r)$ contained in the Hamiltonian (3.4) on the assumption of a conserved vector $\boldsymbol{R}$, and thereby to find an extended Taub–NUT metric. Now, from (3.5) it follows that the vector $\boldsymbol{R}$ is a conserved vector, i.e., $\mathrm{d}\boldsymbol{R}/\mathrm{d}t = 0$, if and only if

$$\kappa = -\frac{r^2 f'(r)}{2f(r)}|\boldsymbol{p}|^2 - \mu^2\frac{r^2 g'(r) f(r)}{2g(r)^2} + \frac{\mu^2}{r}, \tag{3.10}$$

which should give rise to differential equations for $f(r)$ and $g(r)$.

Since (3.10) is an identity in r and $|\boldsymbol{p}|$, the coefficient of $|\boldsymbol{p}|^2$ must vanish, so that $f'(r) = 0$. We thus have

$$f(r) = b = \text{const.} \neq 0. \tag{3.11}$$

Then, eq. (3.10) turns into a differential equation for $g(r)$,

$$\frac{g'(r)}{g(r)^2} + \frac{2(\kappa r - \mu^2)}{\mu^2 b r^3} = 0,$$

which is easily integrated to give

$$g(r) = \frac{br^2}{1 + cr + dr^2}, \quad d\text{: const.}, \tag{3.12}$$

where $c = -2\kappa/\mu^2$. Thus $f(r)$ and $g(r)$ are determined in a simple manner. It is to be noted that along with the functions thus obtained the Hamiltonian takes the form

$$H = \frac{|\boldsymbol{p}|^2}{2b} + \frac{\mu^2}{2b}\left(\frac{1}{r^2} + \frac{c}{r} + d\right),$$

which is an extension of the MIC–Zwanziger Hamiltonian [12–14].

We can take another way to obtain the conserved vector (3.9). Relaxing the condition of conservation, we may think of (3.9) as a conserved vector under the condition that the total energy is conserved. Setting the value of the Hamiltonian H to E, we can rewrite (3.10) in the form

$$\kappa = -r^2 f'(r)E + \frac{\mu^2}{r} + \frac{\mu^2 r^2[g(r)f'(r) - f(r)g'(r)]}{2g(r)^2}. \tag{3.13}$$

In order that the right-hand side of (3.13) is constant in r, the coefficient of E must be constant. Therefore, one can set

$$-r^2 f'(r) = a, \quad a\text{: const.}, \tag{3.14}$$

which is easily integrated to give

$$f(r) \doteq a/r + b, \quad b\text{: const.} \tag{3.15}$$

Then, eq. (3.13) comes out to be a differential equation for $g(r)$,

$$\frac{2}{\mu^2 r^2}\left(\kappa - aE - \frac{\mu^2}{r}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{a/r + b}{g}\right), \tag{3.16}$$

which is solved by

$$g(r) = \frac{ar + br^2}{1 + cr + dr^2}, \quad d\text{: const.}, \tag{3.17}$$

where $c = 2(aE - \kappa)/\mu^2$. In this case, the Runge–Lenz vector takes the form

$$\boldsymbol{R} = \boldsymbol{p} \times \boldsymbol{J} - \kappa \boldsymbol{x}/r, \quad \text{with } \kappa = aE - \tfrac{1}{2}c\mu^2, \tag{3.18}$$

from which we find that the vector $\boldsymbol{R}$ is in fact conserved when E is a constant. It is to be noted that if $a = 0$ eqs. (3.15) and (3.17) reduce to (3.11) and (3.12), respectively. The above discussion results in the following theorem.

Theorem 3.1. *Assume that the reduced Hamiltonian system from the geodesic flow system for a generalized Taub–NUT metric* $\mathrm{d}s_{\mathrm{G}}^2$ *has a conserved Runge–Lenz vector of the form (3.9). Then* $\mathrm{d}s_{\mathrm{G}}^2$ *becomes the extended Taub–NUT metric* $\mathrm{d}s_{\mathrm{K}}^2$ *given by (1.1) with (1.3),*

$$\mathrm{d}s_{\mathrm{K}}^2 = \frac{a + br}{r}[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)] + \frac{r(a + br)}{1 + cr + dr^2}(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2. \tag{3.19}$$

If $ab > 0$ and $c^2 - 4d < 0$ or $c > 0$, $d > 0$, no singularity of the metric appears in $\mathbb{R}^4 - \{0\}$. However, if $ab < 0$, a manifest singularity appears at $r = -a/b$. Singular geodesics approaching $r = -a/b$ will be discussed later.

3.2. TRAJECTORIES OF THE REDUCED SYSTEM

We denote by H_{K} the Hamiltonian (3.4) with $f(r)$ and $g(r)$ given by (3.15) and (3.17), respectively. For the Hamiltonian system $(T^*(\mathbb{R}^3 - \{0\}), \omega, H_{\mathrm{K}})$ the existence of the Runge–Lenz vector in addition to the angular momentum vector has a geometric consequence of interest on trajectories in the base space $\mathbb{R}^3 - \{0\}$. The discussion runs parallel to the case of the Taub–NUT metric [8]. As is easily seen from (3.6), the inner product of $\boldsymbol{J}$ with $\boldsymbol{x}/r$ is given by

$$\boldsymbol{J} \cdot \boldsymbol{x}/r = \mu, \tag{3.20}$$

which means that trajectories lie on a cone with axis $\boldsymbol{J}$. We assume for a while that $\mu \neq 0$. Setting a conserved vector

$$\boldsymbol{N} = \mu \boldsymbol{R} + \kappa \boldsymbol{J} \tag{3.21}$$

with κ given in (3.18), we have

$$N \cdot x = \mu(|J|^2 - \mu^2), \tag{3.22}$$

which implies that trajectories lie also in the plane perpendicular to N. Hence, it turns out that trajectories are conic sections. The form of the trajectory depends on the inclination of the plane. Since

$$N \cdot J = \kappa(|J|^2 - \mu^2),$$

$$|N|^2 = [\mu^2(2bE - d\mu^2) + \kappa^2](|J|^2 - \mu^2), \tag{3.23}$$

comparison of the plane's inclination angle with the cone's opening angle shows that the trajectories are ellipses, parabolae, or hyperbolae, according to whether $2bE - d\mu^2$ is negative, zero, or positive, as long as $\kappa \neq 0$ and $|J| \neq |\mu|$. If $\kappa = 0$ and $|J| \neq |\mu|$, the trajectories are all hyperbolae because of $N \cdot J = 0$. If $|J| = |\mu|$, the cone collapses to a line and no conic sections appear. This case is to be discussed below.

If $\mu = 0$, the cone becomes a plane perpendicular to J. We are then left with the same setting as the Kepler problem. Hence, trajectories are expressed as conic sections if $J \neq 0$, as is well known.

We point out here that the integrals J and R satisfy the following Poisson bracket relations with respect to the symplectic form (3.3):

$$\{J_\lambda, J_\nu\} = \sum \epsilon_{\lambda\nu\eta} J_\eta,$$

$$\{R_\lambda, J_\nu\} = \sum \epsilon_{\lambda\nu\eta} R_\eta,$$

$$\{R_\lambda, R_\nu\} = (d\mu^2 - 2bH) \sum \epsilon_{\lambda\nu\eta} J_\eta, \tag{3.24}$$

as is expected from the same relations known for the original Taub–NUT metric [4,5]. Thus we recognize that the Hamiltonian system $(T^*(\mathbb{R}^4 - \{0\}), \omega, H_K)$ admits the same symmetry as the Kepler problem does.

We proceed to discuss singular trajectories. The singularity of the Hamiltonian vector field for H_K appears at $r = -a/b$ if $a/b < 0$, as is known from (3.5) with (3.15) and (3.17). We are to ask if trajectories reach the singular sphere $S = \{x \in \mathbb{R}^3 - \{0\}\,;\, r = -a/b > 0\}$. The conservation of energy

$$E = \frac{r}{2(a + br)}|p|^2 + \frac{\mu^2(1 + cr + dr^2)}{2r(a + br)} \tag{3.25}$$

implies that if $p \neq 0$ then r is not allowed to tend to $-a/b$. If the flow of X_H should be regularized as r tends to $-a/b$, p must go to zero and the quadratic polynomial $1 + cr + dr^2$ must have a factor $a + br$, that is, the relation

$$b^2 - abc + a^2 d = 0 \tag{3.26}$$

must hold. If $p \to 0$ along a trajectory, we see from the definition (3.6) that $|J| \to |\mu|$ along the trajectory. Then, the conservation of the angular momentum implies that $|\boldsymbol{J}| = |\mu|$ along the trajectory. Since $|\boldsymbol{J}|^2 = |\boldsymbol{x} \times \boldsymbol{p}|^2 + |\mu|^2$, one has $\boldsymbol{x} \times \boldsymbol{p} = 0$. Thus the trajectory reaching the singular sphere should be radial. Therefore, we can pursue the trajectory in the two-dimensional phase space with variables (r, p), $p := |\boldsymbol{p}|$, using the energy conservation, which can be put in the form

$$r^2 p^2 = (2bE - d\mu^2)r^2 + (2aE - c\mu^2)r - \mu^2. \tag{3.27}$$

The curve of this equation is symmetric with respect to the r-axis, so that the Hamiltonian flow approaching the r-axis can be continued along the curve (3.27) after the flow reaches the r-axis. In other words, a particle going radially to the singular sphere S reaches S in a finite time, and follows the same trajectory backward, just after reaching S. If $2bE - d\mu^2 < 0$, trajectories going outward (in the case of $r > -a/b$) will have a turning point r_{m} at which the right-hand side of (3.27) vanishes. Thus, the radial motion may also be considered as periodic, if $2bE - d\mu^2 < 0$. We have regularized, in this way, singular flows of the Hamiltonian vector field for H_{K}. This regularization, in turn, gives rise to regularization of singular geodesic flows for the extended Taub–NUT metric $\mathrm{d}s_{\mathrm{K}}^2$.

3.3. GEODESICS FOR THE EXTENDED TAUB–NUT METRIC

In the reduced system, we have shown that the bounded trajectories are all periodic. However, this will not necessarily imply that all bounded geodesics for $\mathrm{d}s_{\mathrm{K}}^2$ are periodic as well. We are going to study bounded geodesics for $\mathrm{d}s_{\mathrm{K}}^2$. Getting back to the Lagrangian (3.1), we observe that the variable ϕ is also a cyclic variable, and hence obtain a conserved quantity $p_\phi = \partial L/\partial \dot{\phi}$, which is expressed, on account of (3.2), as $p_\phi = f(r) r^2 \sin^2\theta\, \dot{\phi} + \mu \cos\theta$. In the Hamiltonian formalism, p_ϕ is conjugate to the infinitesimal generator $X = \partial/\partial\phi$; that is, $\iota(X)\omega = -\mathrm{d}p_\phi$. On the other hand, we have already obtained the conserved vector $\boldsymbol{J}$. If we choose the z-axis in the direction of $\boldsymbol{J}$, then for $X = \partial/\partial\phi$, the infinitesimal generator of the rotation about the z-axis, eq. (3.8) becomes $\iota(X)\omega = -\mathrm{d}|\boldsymbol{J}|$. Thus we conclude that

$$f(r) r^2 \sin^2\theta\, \dot{\phi} + \mu\cos\theta = |\boldsymbol{J}|. \tag{3.28}$$

Further, we know already that trajectories lie on the cone defined by (3.20), the half opening angle of which we denote by α_0. This fact implies that the variable θ, the latitudinal angle, is constant during the motion,

$$\dot{\theta} = 0, \qquad \theta = \alpha_0. \tag{3.29}$$

Then, from (3.20) and (3.29), eq. (3.28) becomes

$$f(r) r^2 \dot{\phi} = |\boldsymbol{J}|. \tag{3.30}$$

Taking the conserved quantities μ and $|\boldsymbol{J}|$ into the Lagrangian L, which is also equal to the conserved energy E, we obtain

$$E = \tfrac{1}{2} f(r)\dot{r}^2 + \frac{|\boldsymbol{J}|^2 - \mu^2}{2r^2 f(r)} + \frac{\mu^2}{2g(r)}. \tag{3.31}$$

From (3.30) and (3.31), trajectories in $\mathbb{R}^3 - \{0\}$ should be determined. Introducing the variable $u = 1/r$ and taking ϕ as the parameter describing trajectories, we obtain a differential equation for trajectories,

$$|\boldsymbol{J}|^2 (\mathrm{d}u/\mathrm{d}\phi)^2 = -|\boldsymbol{J}|^2 u^2 + (2aE - c\mu^2)u + 2bE - d\mu^2, \tag{3.32}$$

which will be integrated to give conic sections in $\mathbb{R}^3 - \{0\}$, as is anticipated.

After finding trajectories, conic sections, we can determine geodesics by integrating eq. (3.2) for ψ. We are interested in closed trajectories, ellipses, and ask if one can find closed geodesics for closed trajectories. Let u_1 and u_2 $(u_1 < u_2)$ be two solutions to the quadratic equation obtained by setting the right-hand side of (3.32) equal to zero. Then the increment of ψ after traversing a trajectory is found to be given by

$$\Delta_\psi = 2 \int_{u_1}^{u_2} \frac{\mu(cu + d)}{|\boldsymbol{J}|u^2 \sqrt{(u - u_1)(u_2 - u)}} \mathrm{d}u,$$

which is integrated to give

$$\Delta_\psi = \frac{2\pi\mu}{\sqrt{|2bE - d\mu^2|}} \left(c + \frac{d(2aE - c\mu^2)}{2|2bE - d\mu^2|} \right). \tag{3.33}$$

Note here that $2bE - d\mu^2 < 0$ for bounded trajectories. Now it turns out that if $\Delta_\psi/4\pi$ is a rational number the geodesic is closed. The case of $\mu = 0$ is quite easy. In this case, $\Delta_\psi = 0$ and $\psi = \text{const.}$, so that the closed trajectories can be viewed as closed geodesics when lifted to $\mathbb{R}^4 - \{0\}$.

4. The Riemann curvature tensor

In what follows, we will concentrate on the extended Taub–NUT metric $\mathrm{d}s_\mathrm{K}^2$. First we note that for $b = c = d = 0$, the metric $\mathrm{d}s_\mathrm{K}^2$ is flat because of (2.13a). Further, it is already known that the Taub–NUT metric is an Einstein metric [10]. Hence we wish to ask to what extent the metric $\mathrm{d}s_\mathrm{K}^2$ is extended from the original Taub–NUT metric. A first question we are to ask is when the metric $\mathrm{d}s_\mathrm{K}^2$ is an Einstein metric. We can show the following.

Theorem 4.1. *In order that the extended Taub–NUT metric* $\mathrm{d}s_\mathrm{K}^2$ *given by (3.19) be an Einstein metric, it is necessary and sufficient that the constants a, b, c and d are subject to the constraint (1.4). In addition, the Einstein extended Taub–NUT metric is Ricci flat.*

Proof. From (2.10) it follows that ds_G^2 is Einstein, if and only if

$$A(r)^2 + A'(r)f(r)^{-1/2} + D(r)^2 - A(r)B(r) = 0,$$
$$-A'(r)f(r)^{-1/2} - B(r)^2 - B'(r)f(r)^{-1/2} + 2C(r)D(r) + A(r)B(r) = 0. \tag{4.1}$$

These equations are equivalent to

$$A(r)^2 + A'(r)f(r)^{-1/2} + D(r)^2 - A(r)B(r) = 0,$$
$$A(r)^2 - B(r)^2 - B'(r)f(r)^{-1/2} + 2C(r)D(r) + D(r)^2 = 0. \tag{4.2}$$

On inserting (2.7) into (4.2), one has the following equations for $f(r)$ and $g(r)$, along with $F(r) := r^2 f(r)$:

$$2rg(r)[F(r)F'(r) + rF(r)F''(r) - rF'(r)^2]$$
$$+F(r)g(r)^2 - r^2F(r)F'(r)g'(r) = 0, \tag{4.3}$$
$$[rF'(r)g(r)]^2 - F(r)^2[2r^2g(r)g''(r) + 2rg(r)g'(r) - r^2g'(r)^2]$$
$$+r^2g'(r)g(r)F'(r)F(r) + F(r)g(r)^2[3g(r) - 4F(r)] = 0. \tag{4.4}$$

For the extended Taub–NUT metric, $f(r)$ and $g(r)$ are expressed as in (1.3), or (3.15) and (3.17). Hence, on replacing $f(r)$ and $g(r)$ by those expressions, eq. (4.3) gives rise to an identity in r,

$$(abc + a^2d - 3b^2)r^2 + 2(2abd - b^2c)r^3 = 0,$$

which, in turn, provides the equations for the constants a, b, c, d,

$$abc + a^2d - 3b^2 = 0,$$
$$2abd - b^2c = 0. \tag{4.5}$$

These are equivalent to (1.4) because of $a \neq 0$. The functions $f(r)$ and $g(r)$ given by (1.3) with the conditions (1.4) are shown to satisfy also eq. (4.4).

We are further to show that ds_K^2 with the constraints (1.4) is also Ricci flat. From (1.3) together with (1.4), the functions (2.7) are expressed as

$$A(r) = \frac{a + 2br}{2(a + br)(ar + br^2)^{1/2}},$$
$$B(r) = \frac{a}{2(a + br)(ar + br^2)^{1/2}},$$
$$C(r) = -\frac{a^2 + 4abr + 2b^2r^2}{2a(a + br)(ar + br^2)^{1/2}},$$
$$D(r) = \frac{a}{2(a + br)(ar + br^2)^{1/2}}. \tag{4.6}$$

Then the Ricci tensor given by (2.10) can be shown, after a long and straightforward calculation, to vanish, $R_{ij} = 0$. This completes the proof. □

We proceed to ask if the metric ds_K^2 has a self-dual Riemann curvature tensor. Since the ω^i form an orthonormal basis, the volume element dv on the generalized Taub–NUT space is expressed as

$$dv = \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4, \tag{4.7}$$

and the star operator $*$ is thereby defined to give rise to the formulae

$$*(\omega^1 \wedge \omega^2) = \omega^3 \wedge \omega^4, \qquad *(\omega^1 \wedge \omega^3) = \omega^4 \wedge \omega^2, \qquad *(\omega^1 \wedge \omega^4) = \omega^2 \wedge \omega^3. \tag{4.8}$$

Therefore, in order for $\Omega = (\Omega_i^j)$ given in (2.9) to be self-dual, $*\Omega = \Omega$, it is necessary and sufficient that

$$\begin{aligned} A(r)^2 + A'(r) f(r)^{-1/2} &= D(r)[A(r) - B(r)], \\ B(r)^2 + B'(r) f(r)^{-1/2} &= 2D(r)[B(r) - A(r)], \\ B(r)C(r) + C'(r) f(r)^{-1/2} &= 2C(r)D(r) + A(r)^2 + D(r)^2, \\ A(r)D(r) + D'(r) f(r)^{-1/2} &= D(r)^2 - A(r)B(r). \end{aligned} \tag{4.9}$$

By inserting (2.7) together with (1.3) into the first equation of (4.9), we obtain an identity in r,

$$2ab(1 + cr + dr^2)^{3/2} = (a + br)(ac + (bc + 2ad)r + 2bdr^2),$$

from which it turns out that constants a, b, c, d should be subject to the constraint (1.4). Conversely, if the condition (1.4) is satisfied, we have the relations

$$\begin{aligned} A(r)^2 + A'(r) f(r)^{-1/2} &= \frac{ab}{2(a + br)^3} = D(r)[A(r) - B(r)], \\ B(r)^2 + B'(r) f(r)^{-1/2} &= -\frac{ab}{(a + br)^3} = 2D(r)[B(r) - A(r)], \\ B(r)C(r) + C'(r) f(r)^{-1/2} &= -\frac{ab}{(a + br)^3} = 2C(r)D(r) + A(r)^2 + D(r)^2, \\ A(r)D(r) + D'(r) f(r)^{-1/2} &= -\frac{ab}{2(a + br)^3} = D(r)^2 - A(r)B(r). \end{aligned} \tag{4.10}$$

Therefore, (1.4) is also a sufficient condition for self-duality. The curvature form Ω is then expressed as

$$\Omega = E(r) \begin{pmatrix} \Omega_{\mathrm{I}} & \Omega_{\mathrm{II}} \\ -\Omega_{\mathrm{II}} & \Omega_{\mathrm{I}} \end{pmatrix}, \tag{4.11}$$

where

$$E(r) = -\frac{ab}{2(a+br)^3}, \tag{4.12}$$

$$\Omega_{\rm I} = \begin{pmatrix} 0 & \omega^1\wedge\omega^2 + \omega^3\wedge\omega^4 \\ \omega^2\wedge\omega^1 + \omega^4\wedge\omega^3 & 0 \end{pmatrix},$$

$$\Omega_{\rm II} = \begin{pmatrix} \omega^1\wedge\omega^3 + \omega^4\wedge\omega^2 & 2\omega^4\wedge\omega^1 + 2\omega^3\wedge\omega^2 \\ 2\omega^4\wedge\omega^1 + 2\omega^3\wedge\omega^2 & \omega^2\wedge\omega^4 + \omega^3\wedge\omega^1 \end{pmatrix}.$$

The above discussion results in the following theorem.

Theorem 4.2. *In order that the extended Taub–NUT metric ds_K^2 has a self-dual Riemann curvature form, it is necessary and sufficient that the constants a, b, c, d satisfy the constraint (1.4).*

From theorems 4.1 and 4.2 it turns out that for the extended Taub–NUT metric ds_K^2 the following two are equivalent: (1) ds_K^2 has a self-dual Riemann curvature form, and (2) ds_K^2 is an Einstein metric.

Remarks. As was mentioned in the introduction, if a, b, c, d satisfy the constraint (1.4), the extended Taub–NUT metric coincides with the original Taub–NUT metric up to a constant factor on setting $4m = a/b$. In this case, we come to the self-duality of the Taub–NUT metric [10]. As is easily shown, if $*\Omega = \Omega$, one has $R_{ij} = 0$. Thus we verify the latter part of theorem 4.1 again. Further, since condition (3.26) is satisfied by (1.4), singular geodesics for the Taub–NUT metric can be regularized.

5. The Weyl curvature tensor

In this section, we are to ask if the Weyl curvature tensor of the extended Taub–NUT metric is self-dual, in order to know to what extent the extended Taub–NUT metric is actually extended.

For the Weyl curvature tensor given by (2.16), we define a two-form by

$$W_{ij} = \frac{1}{2}\sum_{k,\ell} C^i_{jk\ell}\,\omega^k\wedge\omega^\ell, \tag{5.1}$$

which turns out to be expressed as

$$W_{ij} = \Omega_{ij} - \tfrac{1}{2}(R_{ii} + R_{jj})\omega^i\wedge\omega^j + \tfrac{1}{6}R\omega^i\wedge\omega^j, \tag{5.2}$$

where we have set $\Omega^i_j = \Omega_{ij}$ because of the identification of the tangent bundle with the cotangent bundle by use of the Riemannian metric. Then, at every point of the manifold, the Weyl tensor is thought of as a linear transformation of the space of two-forms $\Lambda^2 := \wedge^2 T^*(\mathbb{R}^4 - \{0\})$. One can break up Λ^2 into self-dual and anti-self-dual parts with respect to the star operator $*$, $\Lambda^2 = \Lambda_+ \oplus \Lambda_-$, where Λ_+ and Λ_- are eigenspaces corresponding to the eigenvalues of the $*$, $+1$ and -1, respectively. According to this decomposition, a basis of Λ^2 can be taken to be

$$\begin{aligned} &\omega^1 \wedge \omega^2 + \omega^3 \wedge \omega^4, \quad \omega^1 \wedge \omega^3 + \omega^4 \wedge \omega^2, \quad \omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^3, \\ &\omega^1 \wedge \omega^2 - \omega^3 \wedge \omega^4, \quad \omega^1 \wedge \omega^3 - \omega^4 \wedge \omega^2, \quad \omega^1 \wedge \omega^4 - \omega^2 \wedge \omega^3, \end{aligned} \tag{5.3}$$

where the two-forms in the first row are in Λ_+ and those in the second row in Λ_-. With respect to the above basis, the representation matrix W of the linear transformation defined by (5.1) is known to take the block diagonal form

$$W = \begin{pmatrix} W^+ & 0 \\ 0 & W^- \end{pmatrix}, \tag{5.4}$$

where W^+ and W^- are 3×3 matrices representing the induced linear transformation of the invariant subspaces Λ_+ and Λ_-, respectively.

To express the matrices W^+ and W^-, we introduce the notation

$$\begin{aligned} h_1 &= A(r)^2 + A'(r)f(r)^{-1/2}, & h_2 &= D(r)[A(r) - B(r)], \\ h_3 &= B(r)^2 + B'(r)f(r)^{-1/2}, & h_4 &= B(r)C(r) + C'(r)f(r)^{-1/2}, \\ h_5 &= 2C(r)D(r) + A(r)^2 + D(r)^2, & h_6 &= A(r)D(r) + D'(r)f(r)^{-1/2}, \\ h_7 &= A(r)B(r) - D(r)^2. \end{aligned} \tag{5.5}$$

These functions are coefficients appearing in the curvature forms given by (2.9), and subject to the relations

$$h_2 = -h_6, \qquad h_4 = -2h_2. \tag{5.6}$$

Then, after a calculation, we find that

$$\begin{aligned} W^+ &= \tfrac{1}{6}(h_1 + 6h_2 - h_3 - h_5 + h_7)W_0, \\ W^- &= \tfrac{1}{6}(h_1 - 6h_2 - h_3 - h_5 + h_7)W_0, \end{aligned} \tag{5.7}$$

where

$$W_0 = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & 2 \end{pmatrix}.$$

Thus we obtain the following proposition.

Proposition 5.1. *A generalized Taub–NUT metric* $\mathrm{d}s_G^2$ *is conformally self-dual or conformally anti-self-dual, i.e.,* $W^- = 0$ *or* $W^+ = 0$, *according to whether*

$$h_1 - 6h_2 - h_3 - h_5 + h_7 = 0, \tag{5.8a}$$

or

$$h_1 + 6h_2 - h_3 - h_5 + h_7 = 0. \tag{5.8b}$$

Before going into the self-duality question, we reconsider the conformal flatness question for the generalized Taub–NUT metric. By definition, the generalized Taub–NUT metric is conformally flat, if and only if $W^+ = W^- = 0$. Thus the metric ds_G^2 is conformally flat, if and only if $h_2 = 0$ and $h_1 - h_3 - h_5 + h_7 = 0$. From $h_2 = 0$, one has

$$\frac{d}{dr} \log(r^2 f(r)) = \frac{d}{dr} \log g(r),$$

which is integrated to give $r^2 f(r) = c_0 g(r)$ with a constant c_0. It then follows from (2.7) that $A(r) = B(r)$. Thus one has $h_1 = h_3$ from (5.5), so that the remaining condition for conformal flatness reduces to $h_5 = h_7$. Hence, from (5.5) with $r^2 f(r) = c_0 g(r)$ one finds that $c_0 = 1$. Thus we come to the necessary and sufficient condition $r^2 f(r) = g(r)$ for conformal flatness, as in proprosition 2.2.

In the remainder of this section, we concentrate on the conformal (anti-)self-duality of the extended Taub–NUT metric given by (1.3). The condition (5.8a) is put in the form

$$\begin{aligned} &A'(r) f(r)^{-1/2} - 6[A(r) - B(r)]D(r) - B(r)^2 \\ &\quad -B'(r) f(r)^{-1/2} - 2C(r)D(r) - 2D(r)^2 + A(r)B(r) = 0. \end{aligned} \tag{5.9}$$

After a straightforward and lengthy calculation with (3.15) and (3.17), one obtains an identity in r,

$$(c + 2dr)(2\sqrt{1 + cr + dr^2} - 2 - cr) = 0. \tag{5.10}$$

In a similar manner, the conformal anti-self-duality condition (5.8b) can be brought into the form

$$(c + 2dr)(2\sqrt{1 + cr + dr^2} + 2 + cr) = 0. \tag{5.11}$$

Equations (5.10) and (5.11) give rise to an equation for the constants a, b, c, d, respectively. Therefore, if $2 + cr > 0$, eq. (5.10) results in

$$c = d = 0, \tag{5.12}$$

and/or

$$d = c^2/4. \tag{5.13}$$

In the case of (5.12), we obtain the relation $g(r) = r^2 f(r)$, so that the metric ds_K^2 becomes conformally flat from proposition 2.2. In the case of (5.13), we are left with the following theorem.

Theorem 5.2. *In order that the extended Taub–NUT metric* ds_K^2 *with* $2 + cr > 0$ *have a self-dual Weyl tensor or be conformally self-dual, it is necessary and sufficient that the constants c and d satisfy (5.13). In this case, one has*

$$W^+ = \frac{c}{2(a+br)(1+cr/2)^2} W_0, \qquad W^- = 0. \tag{5.14}$$

In the case of $2 + cr > 0$, *eq. (5.11) results in (5.12).*

If it happens that $2 + cr < 0$, we have to consider the anti-self-duality condition. Then the same relations as (5.12) and (5.13) result from (5.11). Thus, in contrast to the above theorem, we have the following.

Theorem 5.3. *The extended Taub–NUT metric with* $2 + cr < 0$ *(i.e.,* $r > 2/|c|$*) has an anti-self-dual Weyl tensor or is conformally anti-self-dual, if and only if the constants c and d satisfy (5.13). In this case, one has*

$$W^+ = 0, \qquad W^- = \frac{c}{2(a+br)(1+cr/2)^2} W_0. \tag{5.15}$$

In the case of $2 + cr < 0$, *the conformal self-duality condition (5.8a) or (5.10) results in (5.12). That is, the metric* ds_K^2 *becomes conformally flat.*

Remark. Since the constraints (1.4) satisfy the condition (5.13), theorems 5.2 and 5.3 show that the Taub–NUT metric is conformally self-dual if $2 + cr > 0$, i.e., $(a+br)/a > 0$ and conformally anti-self-dual if $2+cr < 0$, i.e., $(a+br)/a < 0$. Note also that in this case the factor in (5.14) or (5.15) becomes $ab/(a+br)^3$, as is expected from (4.11) and (4.12).

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